Sync or anti-sync – dynamical pattern selection in coupled self-sustained oscillator systems

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Sync or anti-sync – dynamical pattern selection in coupled self-sustained oscillator systems

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Abstract. The dynamics of similar, self-sustained oscillators coupled by a common platform exhibits fascinating collective behavior. Experiments performed with pendulum clocks and metronomes reported both the absence of synchronization, in-phase synchronization, anti-phase synchronization, beat-death phenomenon, or even chaotic dynamics. Here we present a numerical study on two identical self-sustained oscillators placed on a common movable platform. As order parameter for synchronization we use the Pearson correlation coefficient between the oscillators coordinates. As a function of the relevant physical parameters of this system we reproduce all the experimentally reported dynamics. We provide conditions for obtaining stable and emergent in-phase or anti-phase synchronization.

1. Introduction

Spontaneous synchronization of similar units is a very common form of collective behaviour. No matter if the system is biological, electrical or mechanical, synchronization of oscillatory dynamics will emerge under certain favoring circumstances.[1]. When similar units exhibiting self-sustained periodic oscillations interact, the strength of the interaction will decide whether the system exhibits or not an emerging synchronization. The synchronized state is likely to appear if the strength of interaction exceeds a particular value. This threshold value depends on how different the oscillators are: the more the oscillators are different higher threshold values are necessary to achieve synchronization.

The most simple system that can exhibit spontaneous synchronization is formed by two coupled self-sustained oscillators. The simplest way to imagine such system is by connecting somehow two pendulum clocks as the dutch scientist Christian Huygens did in his legendary experiment. Huygens observed that two pendulum clocks suspended on a common support synchronized in anti-phase, meaning that during their dynamics they had a stable 180^0 phase-difference. He called this phenomena "odd kind of sympathy", and carried out a series of studies to investigate the conditions which lead to the emergence of spontaneous synchronization.[2] This simple phase-locked state of the pendulum clocks, was found to be stable through time until an external force perturbed the system. However, after the perturbation ceased, in around thirty minutes, the system was able to return to this stable, anti-synchronized state. In his letters, he predicted that in-phase motion should also be a stable state of the pendulum system, but later he never mentioned to have observed such state. Strangely though, but even today we do not have a simple criteria determining when two coupled and similar self-sustained oscillators will end up in an in-phase or anti-phase synchronized state.

Recently, the problem of emerging synchronization in coupled mechanical oscillator systems entered in the focus of the scientific community again [3]. Huygens' experiment was reconsidered by several groups using modern experimental techniques. The first who re-examined Huygens two pendulum clocks experiment was Benett and his group [4], reporting somehow similar results. They found that the clocks synchronized in anti-phase when the system parameters were close to the value indicated by Huygens. Stable anti-phase motion was obtained however only for very accurately matching frequencies. When the frequency difference was slightly increased, anti-phase synchronization disappeared, and the clock's swing seemingly in an uncorrelated manner. Huygens had some luck thus with his experiments by choosing two clocks with closely matching natural frequencies. Benett and his group found also a "beat death" phenomenon, when due to the interaction one of the clocks stopped working. Panteleone used metronomes instead of pendulum clocks [5]. His experimental setup consisted of a light platform, placed on two empty soda cans. In this system he observed that for small intrinsic frequency differences, the metronomes were likely to synchronize in-phase with a small phase difference. He explained the appearance of the in-phase synchronization with the presence of large oscillation amplitudes, which disturbs the equilibrium state, for which the anti-phase synchronization would emerge. By adding large damping to the motion of the platform, seemingly the possibility of reaching anti-phase synchronization increases considerably [5]. In response to the experiments carried out by Panteleone, Ulrichs and his group, studied the system with computer simulations [6]. For modeling the dynamics of the coupled system they used the equations of motions derived by Pantaleone. They found that the critical coupling strength for large number of metronomes appears to be almost the same as for only two metronomes. In agreement with the observation of the real metronome system, they reported only in-phase synchronization, and confirmed the absence of the anti-phase synchronization. Czolczynski et. al [7] modeled an array of pendulum clocks hanging from an elastically fixed horizontal beam, and reported many types of collective behavior: in-phase synchronization of the pendulums, synchronization in clusters and anti-phase synchronization in pairs. By performing an energy balance analysis they derived conditions for different synchronization modes for the case of two [9] and many pendulum [8] system. Very recently Boda et. al. [10] performed well-controlled experiments with metronomes placed on a rotating platform, and considered also a realistic mechanical model for this system. Both the experiments and computer simulations for realistically chosen system parameters yield only in-phase synchronization.

The present work adhere to the path drawn by the preceding studies and aims to investigate theoretically the simplest coupled self-sustained oscillator system. The main question we intend to clarify concerns the conditions under which such a system will end up in in-phase or anti-phase synchronized states. The Pearson type correlation between the motion of the two oscillators will be used to quantify the nature of the achieved synchronized state, and the parameter state of the system will be thoroughly mapped. For the sake of simplicity we consider the case of two perfectly identical oscillators coupled through a common freely movable platform. Such system could be approximated either by considering two pendulum clocks hanging on a horizontally movable beam (like in the setup in [7]) or by two metronomes placed on a movable platform (like in the setup from [5, 10]).

2. Ideal oscillators coupled through a common platform

We consider a platform of mass M with two identical mass m attached to it through ideal springs with spring constant k. This system represents two ideal harmonic oscillators with mass

m coupled by the common platform with mass M (Figure 1). We denote by x_1 and x_2 the spring deformation values and x_3 is the absolute coordinate (relative to the chosen inertial Reference frame) of the platform. As a first study, let us assume that friction and driving are absent.

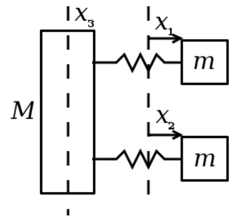

Figure 1. *The considered coupled oscillator system.*

The Lagrange function for this system is

$$L = \frac{1}{2}M\dot{x}_3^2 + \frac{1}{2}m(\dot{x}_1 + \dot{x}_3)^2 + \frac{1}{2}m(\dot{x}_2 + \dot{x}_3)^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2, \tag{1}$$

where the first term is the kinetic energy of the platform, the second and third terms stands for the kinetic energy of the oscillators relative to the chosen inertial reference frame, and the last two terms are the potential energies of the oscillators. The Euler-Lagrange equations of motion are:

$$\begin{aligned} M\ddot{x}_3 &= k(x_1 + x_2) \\ m\ddot{x}_1 &= -kx_1 - m\ddot{x}_3 \\ m\ddot{x}_2 &= -kx_2 - m\ddot{x}_3 \end{aligned} \tag{2}$$

Eliminating the $\ddot{x}_3$ terms, we can derive a system of coupled differential equations yielding the dynamical evolution of the two masses m:

$$\begin{aligned} kx_1 + m(\tfrac{m+M}{2m+M})\ddot{x}_1 - \tfrac{m^2}{2m+M}\ddot{x}_2 = 0 \\ kx_2 + m(\tfrac{m+M}{2m+M})\ddot{x}_2 - \tfrac{m^2}{2m+M}\ddot{x}_1 = 0 \end{aligned} \tag{3}$$

This system allows for an exact analytical solution. Assuming the initial positions of the oscillators $x_1(0) = 1, x_2(0) = a$, and that they are in rest relative to the platform ($\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0$) the exact solutions for $x_1(t)$ and $x_2(t)$ are:

$$\begin{aligned} x_1(t) &= \frac{1}{2}\left((1-a)\cos\frac{\sqrt{k}t}{\sqrt{m}} + (1+a)\cos\frac{k(2m+M)t}{\sqrt{kmM(2m+M)}}\right) \\ x_2(t) &= \frac{1}{2}\left((a-1)\cos\frac{\sqrt{k}t}{\sqrt{m}} + (1+a)\cos\frac{k(2m+M)t}{\sqrt{kmM(2m+M)}}\right) \end{aligned} \tag{4}$$

The Pearson correlation coefficient will be used as a measure of the synchronization level for the two oscillators. It is important to note, that this coefficient will not distinguish between strong (phase-locked) and weak forms of synchronization. This order parameter will be denoted by r and take values between [-1,1]. For a completely in-phase synchronized state $r = 1$ and for a completely anti-phase synchronized state $r = -1$. Mathematically it is defined as:

$$r = \frac{\langle x_1 x_2 \rangle_t - \langle x_1 \rangle_t \langle x_2 \rangle_t}{\sqrt{\langle x_1^2 \rangle_t - \langle x_1 \rangle_t^2}\sqrt{\langle x_2^2 \rangle_t - \langle x_2 \rangle_t^2}} \tag{5}$$

Here we denoted by $\langle x \rangle_t$ the time-average of quantity x. Taking into account that $\langle \cos(\alpha t) \rangle_t = 0$ and $\langle \cos(\alpha t)\cos(\beta t) \rangle_t = 0$ for all $\alpha \neq 0$, $\beta \neq 0$, by simple algebra one gets:

$$r = \frac{2a}{a^2 + 1} \tag{6}$$

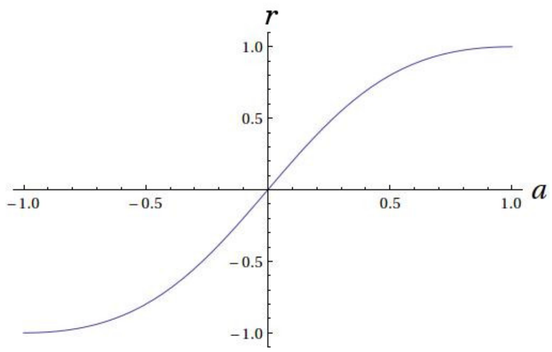

Figure 2. *Pearson correlation of the two oscillators coordinate as a function of the initial position of one of the oscillators ($x_1(0) = 1$, $\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0$ and $x_2(0) = a$, no friction and no driving).*

This means that in this simple system, the synchronization order parameter R (measured through the Pearson correlation) of the oscillators depends only on their initial relative phases and does not depend on any other physical parameters of this system. This universal $r(a)$ curve is plotted in Figure 2. The phase portraits corresponding to different a values are Lissajous curves, and in Figure 3 we illustrate them for some specific a values. The obtained results suggests that complete in-phase synchronization is possible only if the oscillators start with the same initial phases. Similarly, complete anti-phase synchronization will be obtained only if the oscillators are initially in anti-phase. In all other situations only a weak form of synchronization, without phase-locking is possible.

3. Coupled oscillators with damping and driving

Adding damping and driving to the system makes the collective behavior more interesting. In such cases strong synchronization (through phase-locking) can be observed under more general conditions. The equations of motion can be derived from the ones given in (2). Adding friction and driving terms, the equations of motion becomes:

$$\begin{aligned} M\ddot{x}_3 &= k(x_1 + x_2) + C_1(\dot{x}_1 + \dot{x}_2) - C_0\dot{x}_3 - F_1 - F_2 \\ m\ddot{x}_1 &= -kx_1 - m\ddot{x}_3 - C_1\dot{x}_1 + F_1 \\ m\ddot{x}_2 &= -kx_2 - m\ddot{x}_3 - C_1\dot{x}_2 + F_2 \end{aligned} \tag{7}$$

C_0 and C_1 are friction coefficients for the motion of the platform and of the oscillators, respectively, and F_1 and F_2 denote driving forces applied to each of the two masses m. A pulse-like driving acting at the moment when $x_i = 0$ ($i \in \{1, 2\}$) in the direction of the oscillators

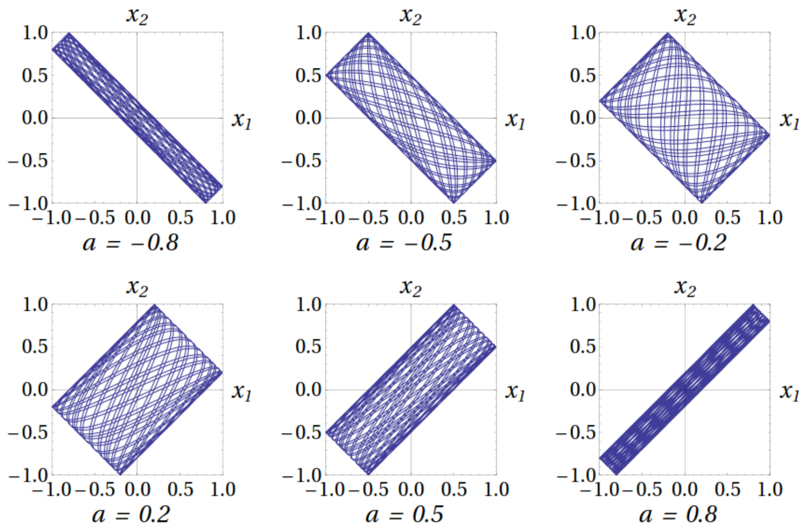

Figure 3. *Phase portraits $x_1(t), x_2(t)$ of two identical oscillators for various $x_2(0) = a$ values. ($x_1(0) = 1$, $\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0$, no friction and no driving).*

motion is used. This is similar with the driving experienced in pendulum clocks or metronomes. We assume thus:

$$F_i = P\dot{x}_i\delta(x_i) \tag{8}$$

P is a coefficient, which characterizes the strength of the given pulses and $\delta(x)$ is the Dirac function. The term $\dot{x}_i$ is needed in order to ensure a constant momentum input, independently of the metronomes' amplitude. It also insures that the excitation is given in the good direction (direction of the motion). It is easy to see that the total momentum transferred, M_{trans}, to the metronomes in a half period ($T/2$) is always P:

$$M_{trans} = \int_t^{t+T/2} P\delta(x_i)\dot{x}_i dt = \int_{-x_{max}}^{x_{max}} P\delta(x_i)dx_i = P \tag{9}$$

The equations of motion with damping and driving cannot be solved analytically, we consider thus a numerical integration by using the Mathematica software. Since the Dirac delta function is not suitable for use with numerical methods, we approximated it with a steeply decaying exponential of the form: $F_i = P\dot{x}_i \exp(-10x_i^2)$.

For setting the units for the mass we fix $m = 1$. We also consider $P = 10$, all over the simulations. This later condition together with the fixed initial condition $x_1(0) = 1$ fixes the units for forces and distances. The spring-constant value and the friction coefficients are given through this unit definition. The dynamics of the system is initialized always as $x_1(0) = 1$, $x_2(0) = a$, $\dot{x}_1(0) = 0$ and $\dot{x}_2(0) = 0$.

The numerical integration for the equations of motions (7) exhibits a high dependence on its parameters rather than the initial position of one of the oscillators. For simulation time $t \gg T$ (where T is the natural period of the oscillators alone) the system shows either no synchronization or a stable in-phase or anti-phase synchronization, depending on parameters M, k, C_0, C_1.

Let us fix as a first example $C_0 = C_1 = 1$ and study the behavior of the system as a function of M and k alone. For $k = 1$ we illustrate some phase-portraits in the x_1, x_2 plane. In the case of $M = 2$ for example, the time-evolution of the system for various initial conditions (quantified by the value of a) is sketched in Figure 4. One will observe that regardless of the initial starting condition, the system ends up in a complete anti-phase synchronized state. As another example, in the $M = 1$ case (Figure 5) the system never synchronize. The fact that the behavior of the self-sustained oscillators shows little dependence on the initial position a, allows us to average the correlation coefficient across all values of a ($R = \langle r \rangle_a$) and to get the general behavior of the system in the M, k parameter plane. We have mapped the R averaged correlation for the $M \in [0, 100]$ and $k \in [0, 100]$ interval, and the results are plotted in Figure 6.

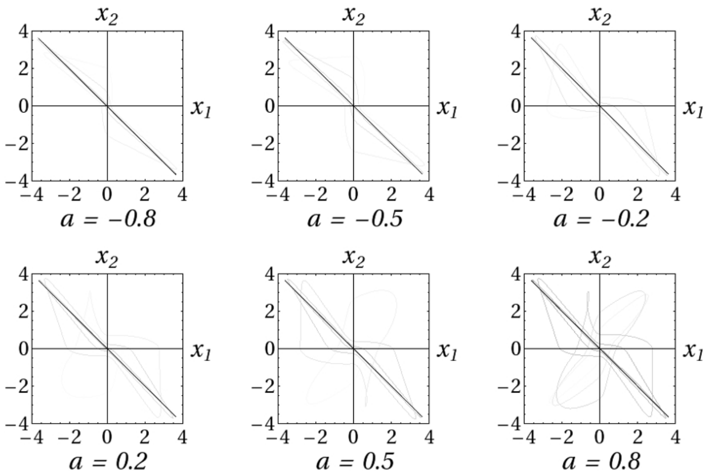

Figure 4. *Phase diagrams for $x_1(t)$, $x_2(t)$ with $M = 2, k = 1$ show convergence to anti-phase synchrony regardless of the initial position a. Darker color denotes greater simulation time. ($C_0 = C_1 = 1$)*

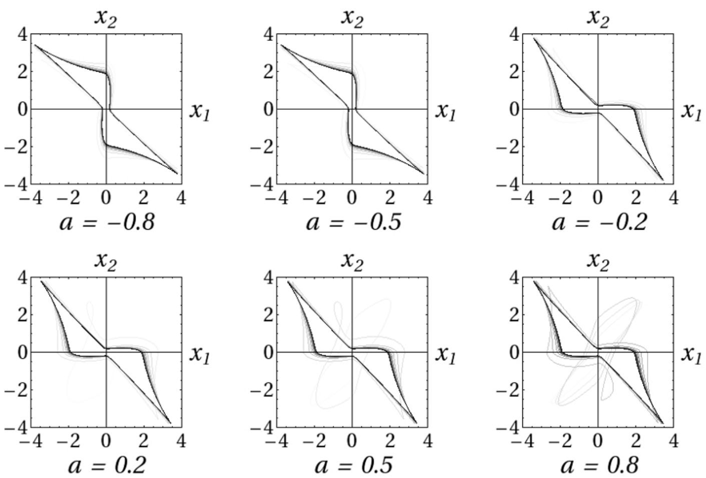

Figure 5. *Phase diagrams for $x_1(t)$, $x_2(t)$ for $M = 1, k = 1$ shows non synchronized state of the system regardless of the initial position a. Darker color denotes greater simulation time. ($C_0 = C_1 = 1$).*

For the chosen friction coefficients $C_0 = C_1 = 1$, one can observe that only the anti-phase synchronized state ($R = -1$) is stable. As a function of M and k, either the phase-locked and complete anti-phase-synchronized state is obtained, or no synchronization is detected ($R = 0$). In the intermediate states $R \in (0, 1)$ the nature of the final stable dynamics depends on the chosen initial condition. As a function of a, either in-phase synchronized states, anti-phase synchronized states or no synchronization is achieved. The $R > 0$ values suggests however that anti-phase synchronization is more often obtained than in-phase synchronization.

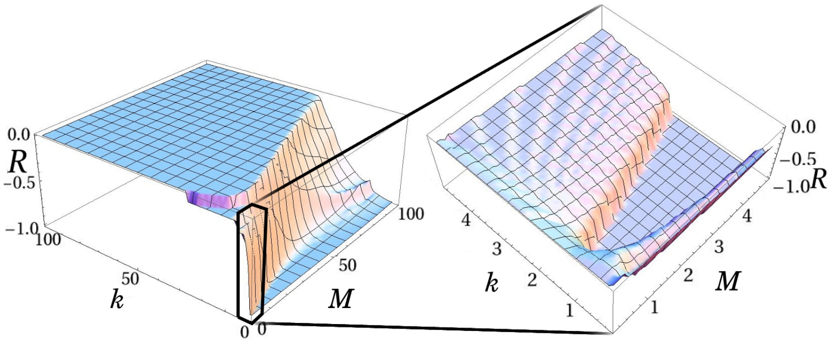

Figure 6. *Averaged correlation, R, as a function of M and k. ($C_0 = C_1 = 1$).*

We turn now our attention to the C_0 and C_1 friction coefficients. The value of the friction coefficient of the platform C_0 does not affect qualitatively the motion of the oscillators very differently than the mass M. By increasing either of them, the oscillation amplitude of the platform will decrease. Changing however the C_1 friction coefficient of the oscillators the picture becomes largely different, and one can achieve also a state of total in-phase synchrony in the system. Increasing the value of the C_1 damping without changing the driving force leads to much lower amplitudes of the oscillations and large numerical errors accumulates in our calculations. Meaningful results for $k = 1$ were obtained thus for friction coefficient values $C_1 < 15$. Numerical results obtained for R as a function of C_1 and M are plotted in Figure 7. This plot suggests that complete in-phase synchrony can be achieved with high values of the C_1 friction values. Seemingly thus the selection of the phase locked complete in-phase or anti-phase synchronized states are mainly governed by the choice of the C_1 parameter. For experimentally reasonable situations with $M > m$, we found that low values of C_1 are favoring the in-phased synchronized states and high values of C_1 will lead to anti-phase synchronized states. This simple picture can be nicely illustrated by considering now a multi-dimensionally averaged correlation across all $M \in [1, 50]$, $k \in [1, 50]$ and $a \in [-1, 1]$ ($R' = \langle r\rangle_{a,M,k}$) and plot it as a function of C_1. The results plotted on Figure 8 shows clearly that there are region of C_1 where the nature of the achieved synchrony (in-phase or anti-phase) is independent of any other parameters than solely the value of C_1.

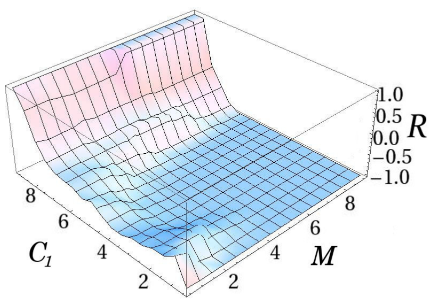

Figure 7. *Averaged correlation, R, as a function of M and C_1. ($C_0 = 1, k = 1$).*

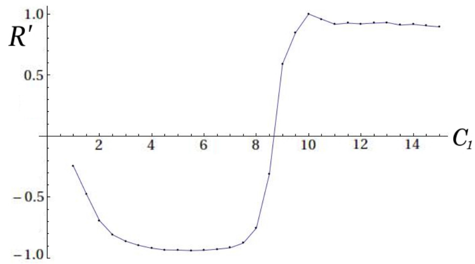

Figure 8. *Correlation averaged across all* $M \in [1, 50]$*,* $k \in [1, 50]$ *and* $a \in [-1, 1]$ *(*$R' = \langle r \rangle_{a,M,k}$*) as a function of* C_1*. (*$C_0 = 1$*).*

4. Conclusions

Spontaneous synchronization in a system composed of two identical self-sustained oscillators coupled through a common platform was investigated. Such systems can be realized experimentally using pendulum clocks suspended on a common beam or metronomes placed on a movable platform. The equations of motions were numerically integrated, and a Pearson type correlation coefficient was used for quantitatively characterizing the emerging collective behavior. Particularly, we were interested in determining the conditions under which stable phase-locked in-phase and anti-phase synchronization emerges. Simulations suggested that depending on the system's parameters, the dynamics of the two oscillators will show either a stable in-phase or anti-phase synchrony, or the system will fail to converge to any type of synchrony at all. We found that spontaneous synchronization is present in a relatively large part of the parameter space. The main parameter that will decide whether the in-phase or the anti-phase synchrony is selected is the ratio of the oscillators friction coefficient and the platforms friction coefficient (C_1/C_0). For $3 < C_1/C_0 < 7$ mainly the anti-phase synchrony is the stable dynamics, while for $C_1/C_0 > 10$ the in-phase synchronized collective behavior is dominant. This selection can be also understood by assuming the convergence to an energy balanced state with zero total momentum, the nature of which is determined by the friction coefficients value, as it was studied in [9]. A simple and elegant explanation for the observed dynamical pattern selection is still lacking however.

Acknowledgments

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