Phase transition in an optimal clusterization model

Zoltán Néda[a],*, Răzvan Florian[b], Mária Ravasz[a], András Libál[c], Géza Györgyi[d]

[a]Department of Physics, Babeş-Bolyai University, str. Kogălniceanu 1, RO-400084 Cluj-Napoca, Romania
[b]Center for Cognitive and Neural Studies, str. Saturn 24, RO-3400 Cluj, Romania
[c]Physics Department, University of Notre Dame, Notre-Dame IN-46556-5670, USA
[d]Department of Theoretical Physics, Eötvös Loránd University, Budapest, Hungary

Abstract

An optimal clusterization model resembling the infinite-range Potts glass-type model with $\pm J$ bonds and unrestricted number of states, $p = N$ is introduced and studied. As a function of the q probability of $+J$ bonds, it is found that the r relative size of the largest cluster, or, coalition, shows a percolation-like transition at $q = \frac{1}{2}$. By a simple renormalization approach and several optimization methods we investigate the $r(q)$ curves for finite system sizes. Non-trivial consequences for social percolation problems are discussed.
© 2005 Elsevier B.V. All rights reserved.

1. Introduction

The Potts glass was originally introduced for studying various non-magnetic random orientational [1] and structural [2] glasses, which do not possess reflection or rotational symmetries. Apart of the specific solid-state and statistical physics applications, the infinite-range (or mean-field) version of the model recently received renewed interest from the view-point of coalition formation phenomenon in sociological systems [3–5]. From this perspective the primary interest is in the ground-state of the infinite-range (or mean-field) p-state Potts glass.

The infinite-range p-state Potts glass is usually defined by the

$$H = -p \sum_{i<j} J_{ij}\delta_{\sigma(i)\sigma(j)} \tag{1}$$

Hamiltonian, where the $\sigma(i)$ Potts states can take the $0, 1, 2, \ldots, p - 1$ values. The sum is extended over all $N(N - 1)/2$ pairs, $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ otherwise. The J_{ij} bonds are randomly distributed quenched variables with J_0/N mean, and the variance is presumed to scale as N^{-1}. The system is non-trivially frustrated and computing the thermodynamic parameters is a complex task. The above model has been extensively

*Corresponding author.
E-mail address: zneda@phys.ubbcluj.ro (Z. Néda).

studied by many authors through different methods [6–14]. Within the replica theory a self-consistent description of the low-temperature glassy phase was obtained [8,10]. For $p > 2$ and low enough temperatures it was found [6,11] that the infinite-range Potts glass is finally always ferromagnetic.

Here we consider a model resembling the infinite-range Potts glass, which can be useful in considering optimal clusterization problems or in understanding coalition formation phenomena in sociological systems. A difference to the Potts glass is that now the variance of the J_{ij} bonds scales as N^{-2}. Also, we consider unrestricted number of Potts states ($p = N$), and limit the study on the ground-state ($T = 0$). The main difference between the original Potts glass and the model studied here is in the degeneracy of configurations. Indeed, a given map of coalitions (clusters) corresponds to many Potts configurations. In our model, in the $N \to \infty$ limit an interesting percolation-like transition is then revealed, which is investigated for finite system sizes by different approximations.

We have to emphasize here that the term "Coalition", frequently used throughout this work should be handled with care. Its more complex classical meaning in politics should not confuse us. In our interpretation a coalition will simply denote a cluster of agents that are all characterized with the same Potts state.

2. The model

In order to describe the process of aggregation, coalition-formation or some optimal clusterization problems in politics, economics or sociological systems we introduce a model similar to the original Potts glass model. In such a system given a set of N actors (analogous to the Potts spins) we define an associated distribution of bilateral propensities towards either cooperation or conflict [3]. The actors might be countries which ally into international coalitions, companies that adopt common standards and strategies, parties that make alliances, individuals which form different interest groups, and so on. The propensities will define the Z_{ij} interactions between the actors. The Z_{ij} bond is positive if there is a tendency towards cooperation and negative if there is a conflicting relation between actor i and j. For simplicity reasons let us assume first that the Z_{ij} links are symmetric ($Z_{ij} = Z_{ji}$), however, later the case without this assumption is also considered. In addition to this, each actor has an $S_i > 0$ weight-factor which characterizes its importance or size in the society. This may be a demographic, economic or military factor, or an aggregate parameter. The question then arises as what kind of coalitions are formed in order to optimally satisfy the conflicting interactions. This coalition structure (Potts variables assignment) is denoted as the optimal coalition state. In particular, we are interested in the size of the largest coalition in the optimal state.

This non-trivial optimization problem can be mathematically formulated resembling a zero-temperature Potts glass type model. To prove this, we define a cost-function, K, (a kind of energy of the system) that is increasing with $S_i S_j |Z_{ij}|$ whenever two conflicting actors (i and j) are in the same coalition or two actors which have a tendency towards collaboration are in different coalition. The cost-function is zero, when no propensity is in conflict with the formed coalitions. The number of possible coalitions is unrestricted (maximal possible number is N), and we denote the coalition in which actor i is by $\sigma(i)$. The cost function then writes as

$$K = -\sum_{i<j} \delta_{\sigma(i)\sigma(j)} Z_{ij} S_i S_j + \frac{1}{2} \sum_{i<j} (Z_{ij} S_i S_j + |Z_{ij} S_i S_j|) . \qquad (2)$$

One immediately realizes that for a given distribution of the Z_{ij} interactions and S_i weight-factors the second term in Eq. (2) is constant (independent of the formed coalitions).

While the function (2) has energy levels equivalent to (1) with $p = N$ and corresponding J_{ij}'s, the degeneracy of the levels makes the models different. The total number of states in the $p = N$ Potts model is obviously $C_{\text{Potts}}(N) = N^N$. Hence the high temperature entropy $\ln C_{\text{Potts}}(N)$ grows faster than linearly in N, that is, the $p = N$ Potts model is super-extensive. Therefore, the number of configurations for the coalition formation model $C(N)$ is bounded from above by N^N, because a given coalition structure corresponds to many a Potts configurations, as explained in A.4. This, however, still leaves the question of extensivity open, so we took a closer look on the number of configurations, see Appendix A. Numerical evaluation, by the method in A.2, of $\ln C(N)$ up to $N = 1000$ shows that it, too, grows faster than a linear function of N. Hence the high

temperature entropy is super-extensive, so we have reasons to believe that traditional methods of statistical mechanics break down.

Returning to our K cost function (2), for $S_i S_j Z_{ij}$ we now introduce the $J_{ij} N$ notation. If Z_{ij} and S_i are independent of N we have that $\langle J_{ij} \rangle$ scales as N^{-1}, and we introduce the notation: $J_0 = N \langle J_{ij} \rangle$. We consider now the practically important case, when the variance of J_{ij} scales as N^{-2}. (An immediate example for this scaling is the simple case when $S_i = S_j = 1$ and Z_{ij} is $+1$ with a probability q and -1 with a probability $1 - q$.) For this choice of disorder, in the finite-p Potts model, the disorder becomes irrelevant in the $N \to \infty$ thermodynamic limit [6,7] for any finite temperatures.

Let us recall that for finite p and a variance of J_{ij} proportional to N^{-1} the Potts glass becomes a ferromagnet for low temperatures [6]. Blind substitution of $p = N$ and a variance of J_{ij} proportional to N^{-2} into the replica free energy of Ref. [6] results for low temperatures again in a ferromagnetic state, since the large p compensates for the smallness of the variance. However, because of the breakdown of extensivity this conclusion should be handled with care. As we shall see below, the coalition formation model does not behave that way.

What we find rather resembles the $p = 2$ case of the Potts-glass [6,7], i.e., the Sherrington–Kirkpatrick model, in that the disorder with variance proportional to N^{-2} turns out irrelevant even for $T \to 0$. Advancing the results discussed later in the paper, while for $J_0 > 0$ the system has minimal cost function when all elements are in the same coalition, for $J_0 < 0$ in the ground-state each element is in a different coalition. As a function of J_0 a transition is obtained. This transition resembles the one in percolation or random graph models. Since the temperature has no role in this phenomenon, we call it geometrical phase transition.

In the present paper we study this geometrical phase transition for finite N values and simple J_{ij} distributions. The finite N case is, however, not as simple as the thermodynamic limit. Frustration effects are important and finding the ground-state is a complex NP hard optimization problem. (It is believed that for large N the number of steps necessary for an algorithm to find an exact optimum must, in general, grow faster than any polynomial in N.) Several methods were used to investigate finite-size behavior in the expected transition. First, a simple renormalization approach was considered. For small systems (up to $N = 10$) an exact enumeration was then used. For larger systems (up to $N = 60$) Monte Carlo type simulated annealing and the recently proposed extremal optimization was applied.

The order parameter considered by us is the r relative size of the largest cluster. In the thermodynamic limit r has the right behavior, for $J_0 < 0$ we get $r = 0$, and for $J_0 > 0$ we obtain $r = 1$. More precisely, r is computed as

$$r(J_0) = \left\langle \overline{max_{(i)} \left\{ \frac{C_x(i, J_0)}{N} \right\}} \right\rangle_x , \tag{3}$$

where $C_x(i, J_0)$ stands for the number of elements in state i for an x realization of the J_{ij} distribution, when $\langle J_{ij} \rangle = J_0 / N$. Since the ground-state might be degenerated (i.e., many possible configurations with the same minimal energy might exist) we make an average over all these states (denoted in (3) by the over-line). $\langle \ldots \rangle_x$ refers then to an ensemble average over J_{ij}.

We focus now on the simplest model in which we expect this transition, i.e., when J_{ij} is a two-valued quenched random variable, $J_{ij} = 1/N$ with probability q and $-1/N$ with probability $1 - q$ (i.e., when $S_i = S_j = 1$ and Z_{ij} is $+1$ with a probability q and -1 with a probability $1 - q$). The distribution function of the J_{ij} values writes as

$$P(J_{ij}) = q\delta(J_{ij} - 1/N) + (1 - q)\delta(J_{ij} + 1/N) , \tag{4}$$

where $\delta(x)$ denotes the Dirac functional. We assumed here that the J_{ij} links are symmetric ($J_{ij} = J_{ji}$). It is immediately realized that for this distribution

$$\langle J_{ij} \rangle = (2q - 1)/N, \quad (\Delta J_{ij})^2 = \frac{4q(1 - q)}{N^2} . \tag{5}$$

In the view of our previous arguments we expect that in the $N \to \infty$ limit the $r(q)$ curves will indicate a geometrical phase-transition at $q = \frac{1}{2}$.

3. Renormalization approach

Our elementary *renormalization* approach estimates in a mean-field manner the new relative size of the largest state, whenever the system size is doubled. We start from a system composed by only two elements (step 1). In the ground-state, the probability to have these two elements in the same Potts state is $q_1 = q$. The relative size of the largest cluster is then $r_1 = q_1 + (1 - q_1)/2$, since the largest cluster will be the total system with probability q_1, and the original half with probability $1 - q_1$. In step 2 we now double the system size by linking through all possible J_{ij} connections two previous configurations (A and B) with maximal relative size r_1, each of them having two elements. Then, we reduce the four J_{ij} connections between the elements of A and B to a single one, and transform the system into a configuration similar to the one from step 1. This procedure is summarized in Fig. 1.

The new link will be positive (+1) with probability $q_2 = q_1^4 + 4q_1^3(1 - q_1) + 3q_1^2(1 - q_1)^2$, and the new relative size of the largest state is $r_2 = q_2 + (1 - q_2)r_1/2$. The factor 3 from the last term in the expression of q_2 results by considering the new link positive with $\frac{1}{2}$ probability, whenever there are two positive and two negative links (six possible realizations in total). This doubling procedure is then recursively repeated, leading to the simple renormalization equations

$$q_{k+1} = q_k^4 + 4q_k^3(1 - q_k) + 3q_k^2(1 - q_k)^2 , \tag{6}$$

$$r_{k+1} = q_{k+1} + (1 - q_{k+1})\frac{r_k}{2} . \tag{7}$$

The size of the system after k steps is $N = 2^k$.

On the [0, 1] interval, iteration (6) has two stable fix-points: 0 and 1. There is also an unstable fix-point $q = \frac{1}{2}$. Starting the iteration from $q \in [0, \frac{1}{2})$ we get $\lim_{k \to \infty} q_k = 0$ and $\lim_{k \to \infty} r_k = 0$. Choosing $q \in (\frac{1}{2}, 1]$ we get $\lim_{k \to \infty} q_k = 1$ and $\lim_{k \to \infty} r_k = 1$. These results suggests that in an infinite system we have two distinct phases separated by $q_0 = \frac{1}{2}$, as expected. In phase I the r order parameter converges to 0, and in phase II r converges to 1. We get thus the expected percolation-like transition as a function of q.

Using Eqs. (6)–(7) we can also easily plot the $r(q)$ curves for different system sizes. Results in this sense are presented in Fig. 2. These results support our previous arguments.

4. Exact enumeration

For small system sizes ($N \leqslant 10$) exact enumeration is possible. This means that one can computationally map the whole phase-space (all $\sigma(i)$ realizations) for a generated J_{ij} configuration and determine the minimum energy states. Moreover, for $N \leqslant 7$ it was also possible to map all J_{ij} configurations as well. Our results up to $N = 7$ are thus exact. In the $7 < N \leqslant 10$ interval, although the minimum energy states are exactly found, due to
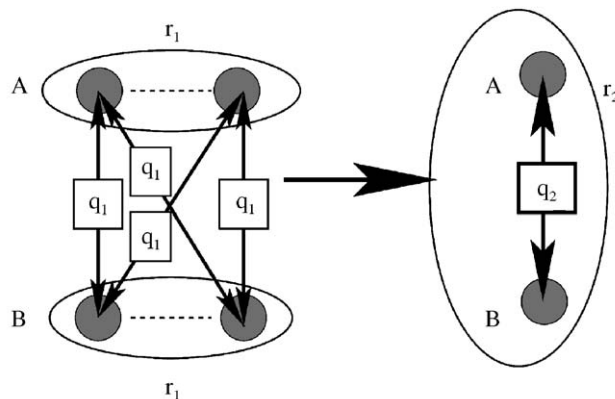

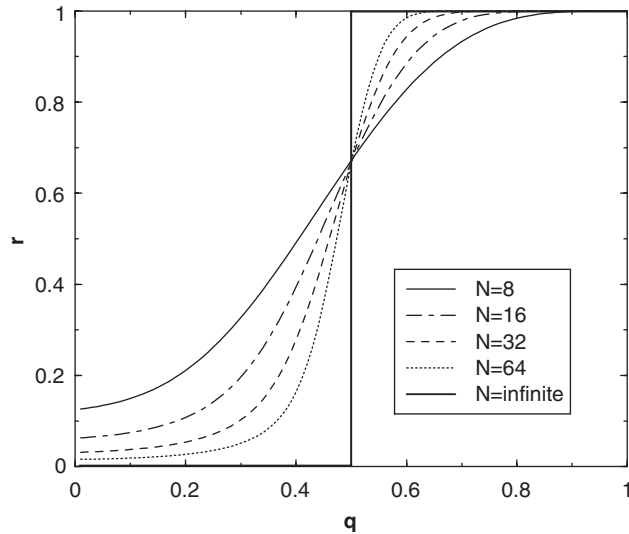Fig. 1. Schematics of the renormalization approach.
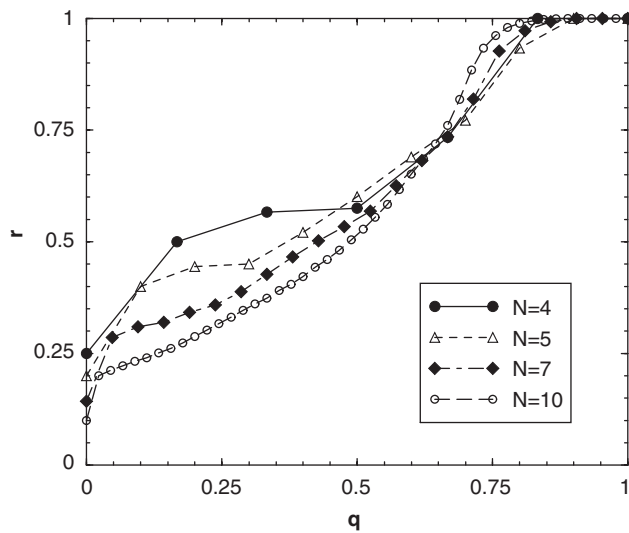
Fig. 2. Renormalization results.

Fig. 3. Exact enumeration results for small systems.

greatly increased computational time and memory needed, it was possible to generate only a reasonable ensemble average for J_{ij} (5000 configurations). Results are plotted in Fig. 3.

We performed this exact enumeration with two purposes. First, we checked the trends of the $r(q)$ curves as a function of increasing system size. Secondly, these results offer a good "standard" for our less rigorous Monte Carlo type optimization methods, used for larger system sizes. As the results in Fig. 3 shows the $r(q)$ curves have a similar trend as those suggested by our renormalization approach, i.e., as the system size increases we find increasing slopes for $r(q)$ around a non-trivial q value.

By exact enumeration we have also studied the degeneration of the ground state. For a given J_{ij} bond-configuration, many different coalition structure can have the same ground-state energy. We can define thus a w degeneration level for each ground-state, and for a fixed q value we can calculate the $\langle w \rangle$ ensemble average over all bond configurations. Different ground-states might be characterized by different r values, as well. For a given bond configuration the difference between the maximal r value (r_{max}) and the minimal one (r_{min}) will
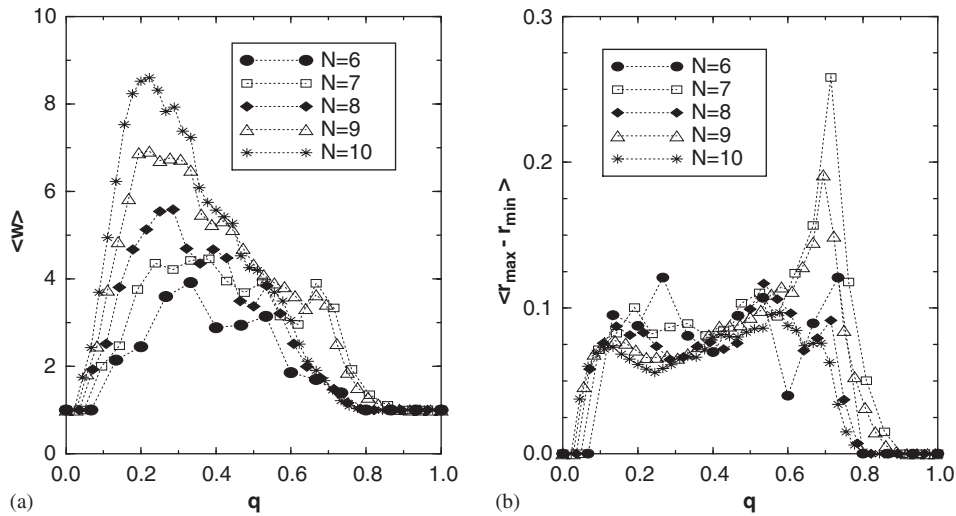
Fig. 4. Average degeneration level of the ground-state: (a) and difference between the maximal and minimal r value for different coalition structures in the ground state (b), both as a function of the q probability of $+$ interactions between the elements.

characterize the maximal possible deviation in the order parameter. An ensemble average over this quantity ($\langle r_{max} - r_{min} \rangle$) will give information about the maximal possible changes in the r order parameter, if another equally optimal coalition structure is chosen. The values of $\langle w \rangle$ and $\langle r_{max} - r_{min} \rangle$ were calculated as a function of the q parameter. The obtained results are presented in Fig. 4. It is interesting to note that the average degeneration of the ground-state is peaking always below the $q_0 = \frac{1}{2}$ percolation threshold (Fig. 4a). The value of $\langle w \rangle$ is increasing with the system size, and the maxima is obtained for smaller and smaller q values. The value of $\langle r_{max} - r_{min} \rangle$ shows a different trend, having a converging tendency with increasing system size and usually the maximal values are obtained for $q > q_0$ values.

5. Monte Carlo type optimization

Monte Carlo type optimizations were used for computing the ground-state of larger systems. We considered both the classical simulated annealing [15] and the recently proposed extremal optimization method [16]. Both approaches are rather time-consuming and the necessary computational time increases sharply with system size. Our computational resources allowed us to study systems with sizes up to $N = 60$.

Simulated annealing has been implemented in the standard fashion [15]. For the extremal optimization method we generalized the originally proposed method [16] by considering a two-step algorithm. In the first step we performed the usual optimization after the energies of the elements. As suggested in Ref. [16] we assigned a given fitness to each Potts element and ranked all the variables according to their fitness. Considering the $P(k) \sim k^{-\tau}$ probability distribution over the rank, k, we then select an element for which the state will be changed. For this first step we found the optimal value of $\tau = 0.25$. In the second step we decide the new state of the chosen element by a similar procedure. For this second step the optimal value of τ proved to be 4.

Simulated annealing and extremal optimization gave identical and practically indistinguishable results. Therefore, in Fig. 5 we plot only the simulated annealing results. The shape of the

$$\Delta r(q) = \sqrt{\langle \overline{r^2(q)} \rangle_x - \langle \overline{r(q)} \rangle_x^2} \tag{8}$$

standard deviation was also computed (Fig. 4b), suggesting a non-trivial peak. In Fig. 5 the curves for $N = 10$, 20, 30 and 40 were obtained with an ensemble average of 5000 realizations, and the results for $N = 60$ with a statistics of 1000 realizations. For $N = 10$ the Monte Carlo type results are in perfect agreement with the ones from exact enumerations (Fig. 5a), giving confidence in the used stochastic simulation methods.
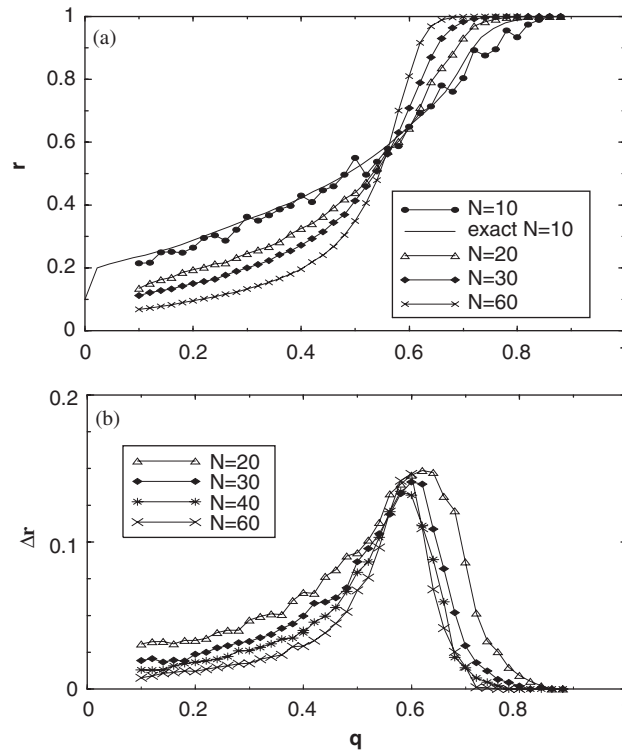
Fig. 5. Monte Carlo optimization results: (a) dependence of the order parameter and (b) standard deviation of the order parameter, both as a function of q. For comparison purposes on (a) the exact enumeration results for $N = 10$ are also shown (continuous line).
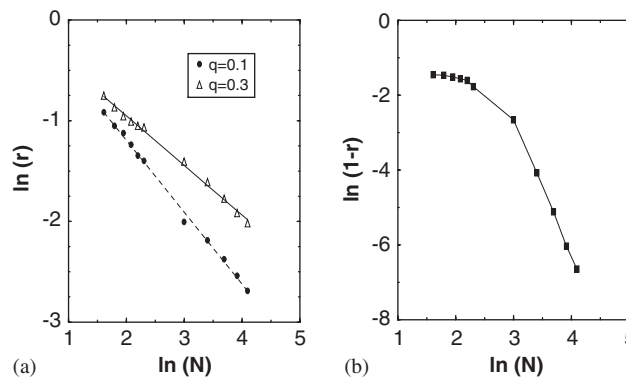

Fig. 6. Finite-size scaling (log–log plots) for (a) $q = 0.1$ and $q = 0.3$ (b) $q = 0.7$. The best-fit lines for (a) have slopes of -0.71 and -0.49, respectively.

Our estimates suggest that extremal optimization was faster by a factor of at least 2, in comparison with simulated annealing. However, we found that extremal optimization is also strongly affected by the increasing system size, and for $N > 60$ we could not get any good statistics in reasonable computational time.

The results plotted in Fig. 5 support the picture of the expected geometrical phase transition. As the system size increases the $r(q)$ curves show a more and more sharper trend in the vicinity of $q = \frac{1}{2}$. Also, the $\Delta r(q)$ standard deviation exhibits a non-trivial peak for $q > q_0 = \frac{1}{2}$, and gets closer to q_0 as the system size increases. By extrapolating the obtained results as a function of N for $q = 0.1$, $q = 0.3$ and $q = 0.7$, one can show that $r \to 0$ as a power-law for $q = 0.1$ and $q = 0.3$, and $r \to 1$ for $q = 0.7$ (Fig. 6). This proves the existence of the presumed phases.

Dropping the symmetry requirement for J_{ij} introduces an extra frustration in the system. While for symmetric J_{ij} only subsets with more than two elements can be frustrated, in the asymmetric case subsets of two elements can become already frustrated. It is interesting to note, however, that the nature of the observed transition is not affected by dropping this symmetry requirement and again, the same geometrical phase transition should appear in $q_0 = \frac{1}{2}$. Up to $N = 10$ we computed the $r(q)$ curves by exact enumeration and for $N = 20$ and 30 we used the extremal optimization method. No important deviations from the symmetric case were found.

6. A more general case

Next, we briefly sketch our results for a more general case, where the S_i factors are also randomly distributed.

Considering a simple uniform distribution of the S_i values on the [0, 1] interval and Z_{ij} distributed according to the (4) distribution, we performed a simulated annealing optimization. Since the variance of J_{ij} scales again like N^{-2}, the transition is naturally expected. For $N = 10, 20$ and 40 results supporting this geometrical phase transition are plotted in Fig. 7.

7. Discussions

The observed phase transition is interesting also from the viewpoint of the much discussed social percolation [17], where the emergence of a giant cluster is observed in many social phenomena. Our simple model suggests that large sociological systems can show tendencies to percolation-like behavior due to the discussed coalition formation or cluster-structure optimization phenomena. If a globally coupled large system
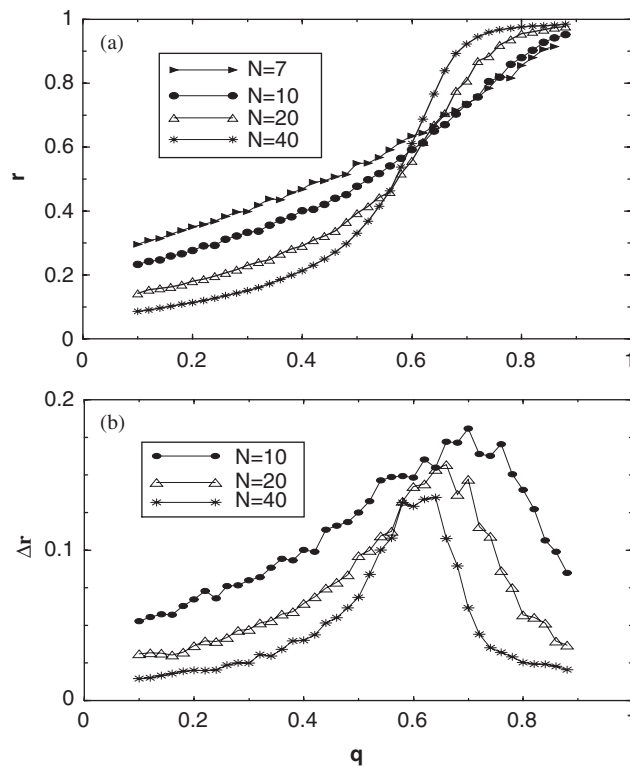

Fig. 7. Simulated annealing results for uniformly distributed S_i values: (a) Dependence of the order parameter and (b) standard deviation of the order parameter, both as a function of q.

has more propensities pointing towards collaboration than conflict, usually a single coalition satisfies optimally the apparently conflicting interactions. On the contrary, when there are more conflicting propensities than collaborative ones, the society will fragment in large number of coalitions, and each element will isolate itself from the others. As expected, this percolation-like behavior is rather smooth for small system sizes. The observed percolation-like behavior is also quite stable relative to the choice of the Z_{ij} propensities and S_i weight-factors.

It is also important to mention that according to the considered model the most unpredictable societies are ones where the number of positive and negative links are roughly the same. From our numerical results one can see that in this case Δr is big, and the value of r is changing strongly with small variations of q. First, this means that the system is very sensitive to the explicit realization of the J_{ij} values. Secondly, as seen in Fig. 4 in this region many ground-state configurations with different r values might co-exist, all of them having the same minimal K cost-function value. Third, a small difference in the measured q value can result in large differences for the expected r values. In these societies statistical methods are useless for predicting the optimal coalition structure.

Predicting the ground-state of large systems with small q values is also difficult. As seen in Fig. 4a the number of optimal coalition structures increases sharply with the system size and peaks for smaller and smaller q values. In our model all these ground-states are equally probable and this high degeneration level makes such a statistical approach ineffective.

The fact that in the ground-state many equally-optimum configurations with quite different maximal cluster sizes are possible might also lead to interesting implications. The existence of some "mixed" states might well be possible, where the system behavior can be described not from a clear coalition structure, but rather from a superposition of many coalition structures.

The model considered by us is of course a very simple one, capturing only a few parameters that are important in understanding social coalition formation. In our model we also neglected the dynamics of the system, and presumed that the system will tend towards the optimal configurations. The system is, however, frustrated, and many configurations with local minimum exist. During its dynamics, the system might get trapped in a local minimum, and the formed coalitions might be the one corresponding to this case, rather than the global optimum case. For implementing a realistic dynamics for coalition formation one should also take in to account that coalitions are not instantaneously and simultaneously formed. Once an agent is assigned to a coalition, it can (and probably will) change its propensities toward other agents. Agent will adjust their propensities according to the already formed coalitions and this feedback presumably reduces the frustration in the system.

All the above limitations of the present model might explain why our results are somehow strange and in disagreement with our everyday experiences regarding the sizes of the coalitions. In real, large sociological systems, we usually observe the presence of several large coalitions with more or less similar sizes. In contrast with this, our model suggests two extreme coalition structure type. We believe that the applicability of our model is limited mainly for the case when an external agent wants to realize an optimal clusterization in the system by taking into account the declared and a priori fixed propensities between the agents and minimizing the (2) cost-function.

In conclusion, in the present study we presented evidences for a geometrical phase transition in the ground-state of an optimal clusterization (or coalition formation) model. This has a cost function similar to the infinite-range, $p = N$, Potts glass where the variance of the bonds scale as N^{-2}. For finite system sizes three different methods were used to approach this NP hard optimization problem, all of them supporting the percolation-like behavior of the largest cluster size as a function of the positive links in the system.

Appendix A. Number of configurations

A.1. Combinatorial formula

In a given coalition structure of N actors there are k_j coalitions of j members each, $j = 1, \ldots N$, such that

$$N = \sum_{j=1}^{N} jk_j .\tag{A.1}$$

While one can place the actors $N!$ number of ways on a coalition map, permutations of actors within a coalition, and permutations of coalitions having the same number of members do not produce different coalition configurations. Hence for a given $\{k_j\}$ coalition structure there are

$$C(N, \{k_j\}) = \frac{N!}{(1!)^{k_1} (2!)^{k_2} \ldots (N!)^{k_N} k_1! k_2! \ldots k_N!}\tag{A.2}$$

configurations. The sought total number of configurations is then given by

$$C(N) = {\sum}' C(N, \{k_j\}) ,\tag{A.3}$$

where $\sum'$ means summation over all $k_j \geqslant 0$ integers with the constraint (A.1).

Formula (A.3) with (A.2), however explicit, is not very practical for computations, because the factorials soon cause overflow, and checking the condition (A.1) also requires extra resources.

A.2. Recursion

In order to compute the total number of configurations we consider the number of configurations $C(N, k)$ with only the number of coalitions

$$k = \sum_{j=1}^{N} k_j ,\tag{A.4}$$

fixed. Hence the total is

$$C(N) = \sum_{k=1}^{N} C(N, k) .\tag{A.5}$$

A coalition configuration with N actors can be obtained from the one with $N - 1$ by either putting the Nth actor into a separate coalition, with him as the only member, thereby augmenting the number of coalitions by 1, or, making him join any of the k existing coalitions. Hence one obtains

$$C(N, k) = kC(N - 1, k) + C(N - 1, k - 1) ,\tag{A.6}$$

$k = 1, \ldots, N$, with the convention $C(N, 0) = C(N, N + 1) = 0$. Note that $C(N, 1) = C(N, N) = 1$. Having computed the $C(N, k)$s, we get the total number of configurations from (A.5).

A.3. Generating function

Out of pure curiosity, we study here the question whether the terms $C(N, k)$ can be obtained from a generating function. For that purpose we first note that the recursion (A.6) is similar to the rule whereby a Pascal triangle is generated, with the difference that we have here an extra k factor.

For a start we can consider

$$F_B(N, x) = (1 + x)^N = \sum_{k=0}^{N} \binom{N}{k} x^k\tag{A.7}$$

as the generating function for the binomial coefficients. If we knew only the Pascal recursion rule for the coefficients, we would get

$$F_B(N, x) = (1 + x) F_B(N - 1, x) \,, \tag{A.8}$$

whence, with $F_B(0, x) = 1$, the explicit formula follows.

Let us turn to the generating function for coalition configurations

$$F(N, x) = \sum_{k=1}^{N} C(N, k) x^k \,. \tag{A.9}$$

Taking into account the coefficient k in the recursion (A.6), we should modify (A.8) accordingly

$$F(N, x) = \left(x \frac{\mathrm{d}}{\mathrm{d}x} + x \right) F(N - 1, x) \,. \tag{A.10}$$

Note that the prefactor is the sum of non-commuting terms. Hence, given $F(1, x) = x$, we get

$$F(N, x) = \left(x \frac{\mathrm{d}}{\mathrm{d}x} + x \right)^N 1 \,. \tag{A.11}$$

The total number of configurations is then obtained as

$$C(N) = F(N, 1) \,. \tag{A.12}$$

A few low order polynomials are

$$F(1, x) = x \,, \tag{A.13}$$
$$F(2, x) = x + x^2 \,, \tag{A.14}$$
$$F(3, x) = x + 3x^2 + x^3 \,, \tag{A.15}$$
$$F(4, x) = x + 7x^2 + 6x^3 + x^4 \,, \tag{A.16}$$
$$F(5, x) = x + 15x^2 + 25x^3 + 10x^4 + x^5 \,, \tag{A.17}$$

and the corresponding total number of configurations are $C(N) = 1, 2, 5, 15, 52$, respectively.

A.4. Comparison with the Potts model

The reasoning above can be used for computing configurations in the $p = N$, infinite range Potts model. A Potts cluster is defined now as the ensemble of spins pointing in the same Potts direction. The number of configurations with k clusters can be simply obtained as

$$C_{\mathrm{Potts}}(N, k) = \frac{N!}{(N - k)!} C(N, k) \,, \tag{A.18}$$

where $C(N, k)$ was introduced in A.2 as the number of configurations with k coalitions. Indeed, one coalition selected as a cluster can point into any of the N different Potts directions, another coalition, selected second, can point into any save the first, and so on, the last picked out of the k coalitions has a Potts degeneracy of $(N - k + 1)$. Thus the known total number of Potts configurations writes, cf. (A.3), as

$$C_{\mathrm{Potts}}(N) = N^N = \sum{}' \frac{N!}{(N - k)!} C(N, \{k_j\}) \,. \tag{A.19}$$

The above formulas demonstrate that the number of configurations in the coalition formation model is significantly less than those in the Potts system.

References

[1] K. Binder, J.D. Reger, Adv. Phys. 41 (1992) 547.
[2] T.R. Kirkpatrick, P.G. Wolynes, Phys. Rev. B 36 (1987) 8552.
[3] R. Axelrod, S. Bennett, Br. J. Political Sci. 23 (1993) 211.
[4] S. Galam, Physica A 230 (1996) 174;
    S. Galam, Physica A 238 (1997) 66.
[5] R. Florian, S. Galam, Eur. Phys. J. B 16 (2000) 189.
[6] D. Elderfield, D. Sherrington, J. Phys. C 16 (1983) L497.
[7] A. Erzan, E.J.S. Lage, J. Phys. C 16 (1983) L555.
[8] D.J. Gross, I. Kanter, H. Sompolinsky, Phys. Rev. Lett. 55 (1985) 304.
[9] T.R. Kirkpatrick, D. Thirumalai, Phys. Rev. B 37 (1988) 5342;
    D. Thirumalai, T.R. Kirkpatrick, Phys. Rev. B 38 (1988) 4881.
[10] G. Cwilich, T.R. Kirkpatrick, J. Phys. A 22 (1989) 4971.
[11] E. De Santes, G. Parisi, F. Ritort, J. Phys. A 28 (1995) 3025.
[12] T. Yokota, Phys. Rev. B 51 (1995) 962.
[13] B.O. Peters, B. Dunrveg, K. Binder, M.D. De Meo, K. Wollmayr, J. Phys. A 29 (1996) 3503.
[14] O. Dillman, W. Janke, K. Binder, J. Stat. Phys. 92 (1998) 57.
[15] S. Kirckpatrick, C.D. Gelatt, M.P. Vecchi, Science 220 (1983) 671.
[16] S. Boettcher, A.G. Percus, Phys. Rev. Lett. 86 (2001) 5211;
    S. Boettcher, A.G. Percus, Artifi. Intell. 119 (2000) 275.
[17] S. Solomon, G. Weisbuch, L. de Arcangellis, N. Jan, D. Stauffer, Physica A 277 (2000) 239.